Assignment 4

This homework is due *Friday*, October 7.

There are total 32 points in this assignment. 28 points is considered 100%. If you go over 28 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations. Bare answers will not earn you much.

This assignment covers section 2.2–2.4 in O'Neill.

(1) (Parts of 2.2.1,2.4.1) For the curve $\alpha(t) = (2t, t^2, t^3/3)$,
   (a) [1pt] find the velocity, speed, and acceleration for arbitrary t, and at $t = 1$.
   (b) [1pt] find the arc length function $s = s(t)$ (based at $t = 0$), and determine the arc length of α from $t = -1$ to $t = 1$,
   (c) [2pt] using s found above, find unit speed reparametrization of α,
   (d) [4pt] compute the Frenet apparatus (κ, τ, T, N, B) of α (either using unit speed reparametrization, or using arbitrary speed formulas),
   (e) [1pt] find limits of T, N, B as $t \to \infty$ and $t \to -\infty$.

(2) [3pt] (2.2.2) Show that a curve that constant speed if and only if its acceleration is everywhere orthogonal to its velocity.

(3) [3pt] (Part of 2.3.1) Show that the unit speed curve

$$\beta(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right)$$

is a circle. (*Hint:* Compute its curvature and torsion.)

(4) [3pt] (2.3.5) If A is the vector field $\tau T + \kappa B$ on a unit speed curve β, show that the Frenet formulas become

$$T' = A \times T,$$
$$N' = A \times N,$$
$$B' = A \times B.$$

— *see next page* —

(5) (Based on 2.3.6)

(a) [2pt] Let β be a unit speed curve. Find a unit speed straight line γ that is a first order approximation of β near $\beta(0)$, i.e. such that

$$\gamma(0) = \beta(0) \text{ and } \gamma'(0) = \beta'(0).$$

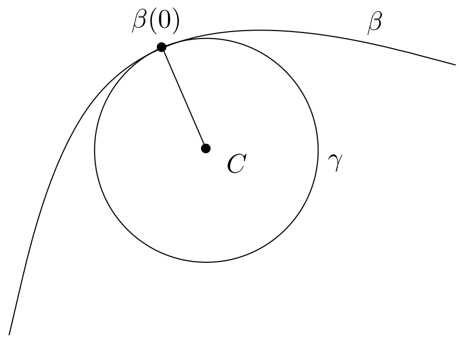
(b) [4pt] A unit speed parametrization of a radius r circle centered at C may be written

$$\gamma(s) = C + E_1 r \cos\frac{s}{r} + E_2 \sin\frac{s}{r}$$

(here $C, E_1, E_2 \in \mathbb{R}^3$), where $E_1 \bullet E_2 = 0$.
If β is a unit speed curve with $\kappa(0) > 0$, find a unit speed circle γ that is a second order approximation of β near $\beta(0)$, i.e. such that

$$\gamma(0) = \beta(0), \gamma'(0) = \beta'(0) \text{ and } \gamma''(0) = \beta''(0).$$

Show that γ lies in the osculating plane of β at $\beta(0)$ and find its center C and radius r. (See Fig 2.13 in Textbook or the figure below.)



COMMENT. It is not hard to show that such a straight line in (a) and such a circle in (b) are unique. The line in (a), as you know, is called the tangent line of β at $\beta(0)$. Circle γ in (b) is called the *osculating circle*, C the *center of curvature* and r the *radius of curvature* of β at $\beta(0)$.

(6) (a) [2pt] (2.3.3) Show that the curve $\alpha(t) = (\cosh t, \sinh t, t)$ has arc length function $s(t) = \sqrt{2}\sinh t$, and find a unit speed reparametrization of α. (Reminder: $\cosh t = (e^t + e^{-t})/2,\ \sinh t = (e^t - e^{-t})/2$.)

(b) [3pt] (2.4.2) Express the curvature and torsion of the curve $\alpha(t) = (\cosh t, \sinh t, t)$ in terms of arc length s measured from $t = 0$.

(7) [3pt] (2.4.4) Show that the curvature of a regular curve in $\mathbb{R}^3$ is given by

$$\kappa^2 \nu^4 = \|\alpha''\|^2 - (d\nu/dt)^2.$$

(*Hint:* Use arbitrary speed formula for κ and expression of $\|u \times v\|^2$ through $\|u\|, \|v\|, u \bullet v$.)